Select Solutions (Line Integrals)

(1f) Compute $\int_C (x^2+y^2+z^2)\,ds$ where C is parametrized by $\vec{r}(t) = \langle t, \cos(2t), \sin(2t)\rangle$ on $0 \le t \le 2\pi$.

Sol: $\int_C f\,ds = \int_{t=a}^{b} f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt$, so we compute

$\vec{r}\,'(t) = \langle 1, -2\sin(2t), 2\cos(2t)\rangle$,

$|\vec{r}\,'(t)| = \sqrt{1+4\sin^2(2t)+4\cos^2(2t)} = \sqrt{1+4} = \sqrt{5}$, and

$f(\vec{r}(t)) = t^2 + \sin^2(2t) + \cos^2(2t) = t^2+1$ Hence

$$\int_C (x^2+y^2+z^2)\,ds = \int_{t=0}^{2\pi} (t^2+1)\sqrt{5}\,dt = \sqrt{5}\left[\tfrac{1}{3}t^3+t\right]_{t=0}^{2\pi} = \sqrt{5}\left(\frac{8\pi^3}{3}+2\pi\right) \boxed{//}$$

(2d) Compute $\int_C y\,dx + z\,dy + x\,dz$ with C parametrized by $\vec{r}(t) = \langle \sqrt{t}, t, t^2\rangle$ on $1 \le t \le 4$.

Sol: Note $\int_C P\,dx + Q\,dy + R\,dz = \int_C \langle P,Q,R\rangle \cdot d\vec{r} = \int_{t=a}^{b} \vec{v}\cdot\vec{r}\,'(t)\,dt$

Now $\vec{v} = \langle y,z,x\rangle = \langle t, t^2, t^{1/2}\rangle$ and $\vec{r}\,'(t) = \langle \tfrac{1}{2}t^{-1/2}, 1, 2t\rangle$, so

$$\int_C y\,dx + z\,dy + x\,dz = \int_{t=1}^{4} \langle t, t^2, t^{1/2}\rangle \cdot \langle \tfrac{1}{2}t^{-1/2}, 1, 2t\rangle\,dt$$

$$= \int_{t=1}^{4} \left(\tfrac{1}{2}t^{1/2} + t^2 + 2t^{3/2}\right)dt$$

$$= \left[\tfrac{1}{3}t^{3/2} + \tfrac{1}{3}t^3 + \tfrac{4}{5}t^{5/2}\right]_{t=1}^{4}$$

$$= \left(\tfrac{1}{3}\cdot 8 + \tfrac{1}{3}\cdot 64 + \tfrac{4}{5}\cdot 32\right) - \left(\tfrac{1}{3}+\tfrac{1}{3}+\tfrac{4}{5}\right)$$

$$= \frac{72}{3} + \frac{128}{5} - \frac{2}{3} - \frac{4}{5}$$

$$= \frac{70}{3} + \frac{124}{5} = \frac{350+372}{15} = \frac{722}{15} \quad \boxed{///}$$

good enough (pointing to $\frac{70}{3} + \frac{124}{5}$)

(3f) Is $\vec{F} = \langle y^2\cos(x) + \cos(y),\ 2y\sin(x) - x\sin(y)\rangle$ conservative? If so, compute a potential function for $\vec{F}$.

Sol: We apply the Conservativity Test.

$$\frac{\partial}{\partial y}\left[y^2\cos(x) + \cos(y)\right] = 2y\cos(x) - \sin(y)$$

$$\|$$

$$\frac{\partial}{\partial x}\left[2y\sin(x) - x\sin(y)\right] = 2y\cos(x) - \sin(y)$$

Pro-tip: That's a joke. Don't @ me.

Hence $\vec{F} = \nabla f$ is Republican. We thus compute a potential function:

$$f(x,y) = \int \frac{\partial f}{\partial x}\,dx = \int\left(y^2\cos(x) + \cos(y)\right)dx = y^2\sin(x) + x\cos(y) + C(y)$$

$$\therefore\ 2y\sin(x) - x\sin(y) = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[y^2\sin(x) + x\cos(y) + C(y)\right] = 2y\sin(x) - x\sin(y) + C'(y)$$

so $C'(y) = 0$ yields $C(y) = \int 0\,dy = D$. Hence

$f(x,y) = y^2\sin(x) + x\cos(y) + D$ is a potential for $\vec{F}$. ☐

(4f) Compute $\int_C \vec{F}\cdot d\vec{r}$ via the Fundamental Theorem for Line Integrals where $\vec{F} = \langle yze^{xz}, e^{xz}, xye^{xz}\rangle$ and C is parametrized by $\vec{r}(t) = \langle t^2+1, t^2-1, t^2-2t\rangle$ with $0 \le t \le 2$.

Sol: First we verify $\vec{F}$ is conservative (letting $\vec{F} = \langle F_x, F_y, F_z\rangle$).

$$\frac{\partial F_x}{\partial y} = \frac{\partial}{\partial y}\left[yze^{xz}\right] = ze^{xz} = \frac{\partial}{\partial x}\left[e^{xz}\right] = \frac{\partial F_y}{\partial x}$$

$$\frac{\partial F_x}{\partial z} = \frac{\partial}{\partial z}\left[yze^{xz}\right] = xyze^{xz} = \frac{\partial}{\partial x}\left[xye^{xz}\right] = \frac{\partial F_z}{\partial x}$$

$$\frac{\partial F_y}{\partial z} = \frac{\partial}{\partial z}\left[e^{xz}\right] = xe^{xz} = \frac{\partial}{\partial y}\left[xye^{xz}\right] = \frac{\partial F_z}{\partial y}$$

Hence $\vec{F} = \nabla f$ is conservative. Now we compute a potential:

$$f(x,y,z) = \int \frac{\partial f}{\partial y}\,dy = \int e^{xz}\,dy = ye^{xz} + C(x,z).$$

$$\therefore yze^{xz} = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[ye^{xz} + C(x,z)\right] = yze^{xz} + C_x(x,z).$$

$\therefore C_x(x,z) = 0$ so $C(x,z) = \int 0\,dx = D(z)$

$\therefore f(x,y,z) = ye^{xz} + D(z)$. Hence

$$xye^{xz} = \frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left[ye^{xz} + D(z)\right] = xye^{xz} + D'(z).$$

$\therefore D'(z) = 0$ so $D(z) = \int 0\,dz = E$.

Hence $f(x,y,z) = ye^{xz}$ is a potential function

Pro-tip: I chose $E=0$ for the constant.

Finally we apply the FTLI. Note that $\vec{r}(0) = \langle 1,-1,0\rangle$ and $\vec{r}(2) = \langle 5,3,0\rangle$. Hence

$$\int_C \vec{F}\cdot d\vec{r} = f(5,3,0) - f(1,-1,0) = 3e^{5\cdot 0} - (-1)e^{1\cdot 0} = 4$$

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